Wake Growth and Collapse in Stratified Flow

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The objective of this research was to obtain data that could be used to predict the effect of stratification on the development of a momentumless wake. Experiments were performed in which a grid was oscillated in a stably-stratified flow to produce the steady-state counterpart of the momentumless wake of a self-propelled vehicle. A pH-sensitive indicator was used to produce a neutrally-buoyant tracer to visualize wake development and subsequent vertical collapse. Measured velocity profiles in the simulated wake indicated that it was nearly momentumless, and vertical temperature surveys revealed the degree of mixing in the wake. The wake growth before and after collapse and the distance to collapse were correlated by using power laws and a theoretical analysis of wake collapse. The scaling relations established for predicting stratified flow wake dimensions revealed that the important parameters were the Froude number, defined as the product of the flow velocity and the Brunt-Vaisala period divided by the initial wake size, and the ratio of the time after wake generation to the Brunt-Vaisala period. Through the use of these parameters, two unique curves were obtained for estimating the horizontal width and vertical height of a wake in a stratified flow.

Nomenclature

A_D = initial wake area
A_w = wake area
α = ambient density gradient, $(1/\rho)\,\partial\rho/\partial z|_a$
β = wake density gradient, $(1/\rho)\,\partial\rho/\partial z|_w$
b_∞ = wake diameter unstratified flow, Eq. (10)
b_v = vertical wake height, Eq. (12)
b_h = horizontal wake width, $t/T < 1.0$, Eq. (16)
b_f = asymptotic vertical wake size, Eq. (13)
b_m = maximum vertical wake size, Eq. (14)
b_c = horizontal wake size for $t/T > 1.0$, Eq. (17)
C_D = drag coefficient of vehicle
D = initial wake diameter
D_B = diameter of vehicle
F = Froude number, uT/D
Fr = Froude number, $u_r\,T/2\pi D$
g = acceleration due to gravity
K = constant of proportionality, $b_\infty/D = K(x/D)^n$
n = exponent, $b_\infty/D = K(x/D)^n$
ρ = ambient density
Ri = Richardson number, Eq. (6)
T = Brunt-Vaisala period, Eq. (2)
t_c = time from the vehicle to wake collapse
t = time after wake generation equal to x/u
u' = characteristic wake turbulent velocity fluctuation, $u' \approx db_\infty/dt$
u = velocity of vehicle
u_r = initial wake turbulent velocity fluctuation
x = distance in the wake from the vehicle equal to ut
z = vertical coordinate in wake from axis

1. Introduction

THE wake of a vehicle travelling through a stably-stratified medium exhibits a behavior considerably different from the more familiar wake of a vehicle moving in an unstratified environment (Sec. 3). In both cases, the wakes grow initially with a circular cross section. However, in the stratified flow case (Sec. 4), mixing in the near wake destroys the ambient temperature gradient so that further downstream buoyancy forces begin to inhibit the vertical growth of the wake. Sometime after generation, the wake, which has a density gradient different from the surroundings, collapses vertically as the fluid seeks its own density level. The decrease in vertical extent of the wake enhances the horizontal growth rate so that eventually a thin and wide wake is produced.

Schooley and Stewart[1] used a self-propelled model in a tank containing stratified water to obtain the first experimental verification of the wake collapse phenomenon and internal wave generation. Stockhausen, Clark, Kennedy and Prych[2-4] studied the wake behind a disk towed through a salt-stratified solution. Other investigators such as Van de Watering,[5,6] Schooley,[7,8] and Sundaram[9] have generated a two-dimensional region of turbulence in a tank containing quiescent stratified water and studied the two-dimensional unsteady equivalent of the three-dimensional wake generated by a moving body. In these cases, the time after turbulence generation is assumed to be equivalent to the residence time in the wake. Wu[10-13] has measured the collapse of a nearly constant-density region imbedded in a quiescent stratified fluid and investigated the internal waves generated as a result. Wessel[14] and Mei[15] produced analyses of Wu's gravity-induced collapse of a constant-density region in a stratified fluid.

Recently Ko[16] has developed an analytical model for predicting the growth and collapse of a turbulent wake in a stratified environment. Ko's analysis, which represents a major step forward, relies heavily on the experimental data discussed previously as well as the turbulence fluctuation measurements of Naudascher,[17] who utilized a jet-disk model in unstratified air to produce a steady-state counterpart of the momentumless wake of a self-propelled body. Other recent work includes that of Pao[18] who measured velocity and concentration fluctuations in the wake of an obstacle towed through a stratified fluid; Schooley and Hughes,[19] who reported measurements and a linear analysis of the internal wave field created by the collapse of a completely-mixed circular wake in stratified water; and Hartman and Lewis,[20] who studied the problem of a partially mixed cylindrical wake.

The objective of the research program described in the present paper is to obtain analytical and experimental data that can be used to determine the effect of various parameters on wake growth and collapse in stratified flow. The experimental approach (Sec. 2) utilizes a stratified flow in which a composite grid is oscillated to produce a steady-state counterpart of the momentumless wake of a self-propelled body. A pH sensitive indicator, in which local color changes are generated at the grid by electrical

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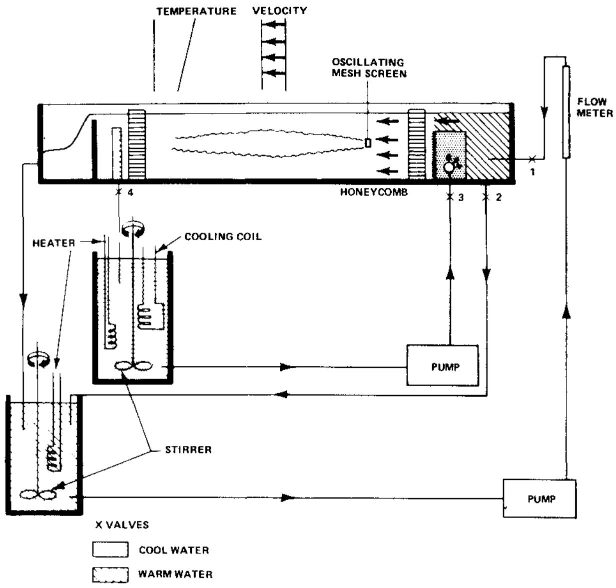

Fig. 1 Schematic arrangement of flow system.

impulses, is used for direct observation of the wake development, diffusion and subsequent vertical collapse.

From the present measurements (Sec. 4.1, 4.3, and 4.4) and analytical efforts (Sec. 4.2) along with those of other investigators, techniques are established (Sec. 4.5) for correlating the experimental data available on wake collapse. The results reveal that the important parameters are the Froude number, defined as the product of the vehicle velocity and the Brunt-Vaisala period divided by the initial wake size, and the ratio of the residence time in the wake to the Brunt-Vaisala period. Through the use of these parameters, two unique curves are obtained for estimating the horizontal width and vertical height of a wake in a stratified flow.

2. Experimental Techniques

A schematic arrangement of the flow system used for the experiments is given in Fig. 1. The tests were performed in a water tank (210 × 10 × 10 cm) in which a stratified flow with a very nearly linear temperature gradient is generated by the mixing of streams of water at different temperatures. The degree of stratification is determined by continually measuring the vertical temperature profile through the use of small thermistor beads that traverse the depth of the water at selected locations at regular intervals.

The velocity of the flow is measured by tracers that are neutrally buoyant at every point in the fluid. This is necessary because the motion of a constant-density dye would be influenced by the density gradient, giving a distorted velocity profile. The technique[21,22] is based on the color change of an indicator solution when the hydrogen ion concentration (pH) is changed at a test point. The solution used consists of water containing 0.01% thymol blue. It is titrated to the end point by adding a small amount of hydrochloric acid and is initially bright orange in color. Fine tungsten wires stretched vertically across the water act as cathodes, and the anode is formed by a stainless steel wire along the tank bottom. Application of brief voltage pulses (90 v) to the two electrodes reduces the hydrogen-ion concentration at the surface of the tungsten wire corresponding to a local increase in pH. Columns of dark tracer fluid are formed at the surface of the tungsten wire and move with the fluid to yield vivid velocity profiles. Typical flow speeds are about 1 cm/sec.

As shown in Fig. 1, a wake analogous to that of a self-propelled body is generated in the water channel test section through the use of a grid oscillating back and forth along the tank axis. The grid which is approximately $\frac{1}{4}$ in. in diameter is pulsed electrically to produce a local pH-color change which traces the wake growth downstream of the grid. Measurements of the wake dimensions in both the horizontal and vertical planes are made at different distances from the grid for both unstratified and stratified flow. In the unstratified case, the wake cross section at any point is circular. When the flow is stratified, buoyancy causes the wake, which initially grows at the same rate in all directions, to collapse in the vertical direction and flatten out horizontally producing a very wide and thin elliptical or rectangular shape.

3. Wake Growth in Unstratified Flow

The growth of a wake downstream of a body travelling through a medium in which there is no temperature gradient has been well documented in the literature. Theoretical analyses along with low-speed tests in water and hypersonic tests in ballistic ranges and wind tunnels have established relations for wake growth of the form

$$b_\infty/D_B = K(C_D\, x/D_B)^n \qquad (1)$$

where b_∞ is the wake width (circular), D_B the diameter of the body, C_D the drag coefficient of the body and x the distance in the wake from the body. For unpropelled bodies, measurements of the turbulent wake growth rates yield values of about unity for K and $\frac{1}{3}$ for the exponent n, in good agreement with theoretical predictions.

For the turbulent wakes of bodies with hydrodynamical self-propulsion, theoretical analyses[23] yield values for n of $\frac{1}{4}$ for plane flow and $\frac{1}{5}$ for axially symmetric flow. At large distances behind the body, the wake may asymptote to a constant size,[17] although this has not been verified experimentally.

Various attempts[5–9] to model a turbulent wake have been reported in which a two-dimensional unsteady flow is produced through the use of paddle-type mixers in a tank containing quiescent water. In these instances, the time t after mixing is equivalent to the distance in the wake x in Eq. (1). It has been observed on the present research program that these measurements can be plotted in the form b_∞/D vs t/D for the cases in which the water is unstratified to give good correlation of the data, with b_∞/D being proportional to $(t/D)^{1/3}$, where D is the initial diameter of the mixed region. The value of $\frac{1}{3}$ for the exponent is high compared with theory[23] which may indicate that the mixed regions generated in the two-dimensional unsteady experiments are not strictly equivalent to those at large distances behind a self-propelled body, the wake of which is expected to grow at a slower rate.

In Fig. 2, it is shown that tests reported by Schooley and Stewart[1] using a self-propelled model yield a wake in unstratified water that can be correlated by a relation $b_\infty/D = 1.0(x/D)^{1/3}$ obtained for hypersonic spheres. The high value of $\frac{1}{3}$ for the exponent indicates that the wake growth in this case is determined largely by the effects of body drag and the propeller. Similarly, experiments by Naudascher,[17] utilizing a jet-disk model in air to produce a steady-state counterpart of the momentumless wake of a self-propelled body, give an initial growth rate of $b_\infty/D = 1.5(x/D)^{2/5}$. This very rapid growth may be due to the influence of the jet for which the exponent n has a value of unity. For distances in the wake greater than about 20 diam, the wake growth in Naudascher's experiment can be represented by $b_\infty/D \approx 1.2(x/D)^{1/4}$ consistent with the theoretical growth rate[23] anticipated for a momentumless wake.

In addition to the interpretation of previous work on wake simulation in terms of power law growth rates, measurements have been made of the wake growth downstream of an oscillating grid in a 1 cm/sec flow of unstratified water. The wake growth observed using the pH-color change tracer is plotted in Fig. 2 from which it is observed that the wake diameter can be expressed by the relation $b_\infty/D \approx 1.3(x/D)^{1/4}$. The wake diameter reaches a size of roughly four times the initial diameter at 100 diam downstream. The measurements indicate that the wake may reach an asymptotic size although further tests are required to

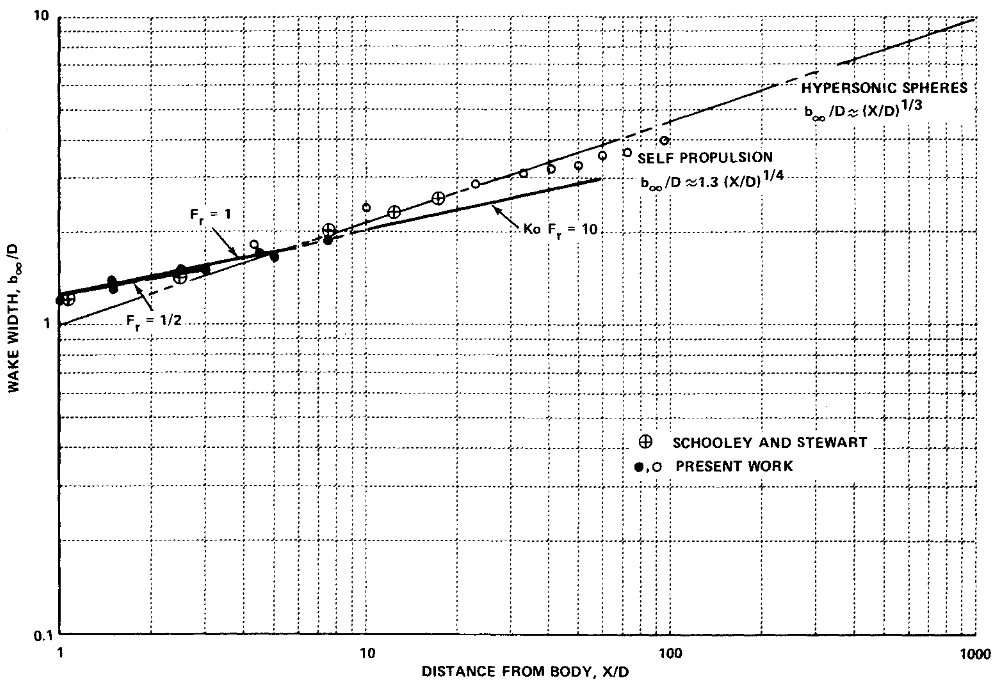

Fig. 2 Growth rates in unstratified flow.

clarify this point. The value $\frac{1}{4}$ for the exponent in the growth law is closer to the value anticipated[23] for the wake growth rate behind a self-propelled vehicle than the value $\frac{1}{3}$ observed in other simulation experiments. Wake velocity profile measurements obtained using the neutrally-buoyant thymol blue tracer indicate little if any velocity defect confirming that the grid wake is nearly momentumless on the average.

Computer results from the theoretical analysis of Ko[16] are reported in the form of several different curves for Froude numbers, Fr of 0.5, 1.0, and 10.0. It is shown in Fig. 2 that these can be expressed by one curve of the form $b_\infty/D = 1.25(x/D)^{0.22}$ during the early growth period before the effects of stratification are felt. This relation compares very well with the equation $b_\infty/D = 1.3(x/D)^{0.25}$ observed in the present experiments for unstratified flow.

The energy of the turbulence generated in the wake by the oscillating grid can be estimated by using the oft-verified[23] result $u' \approx db_\infty/dt$, where u' is a characteristic turbulent velocity fluctuation, and b_∞ is the width of the wake. For the growth rate observed in the experiments, i.e., $b_\infty/D = 1.3(x/D)^{1/4}$ this yields $u'/u = 0.325(x/D)^{-3/4}$ for the rate at which the turbulence intensity decays downstream of the oscillating grid. In Fig. 3 this relation for u' is shown to agree remarkably well with Naudascher's[17] hot wire anemometer measurements in air of the turbulence intensities u', v', w' on the axis behind a jet-disk model which is used to simulate the momentumless wake of a self-propelled body. This agreement confirms the hypothesis that the energy of the turbulence at different positions in the wake can be estimated from measurements of the rate at which the wake dimensions change with time, and that the turbulence decay rate can be expressed by a straightforward power law. It will be shown in the next section that the wake energy represented by the square of the turbulence velocity plays an important role in determining the collapse of the wake when the flow is stratified.

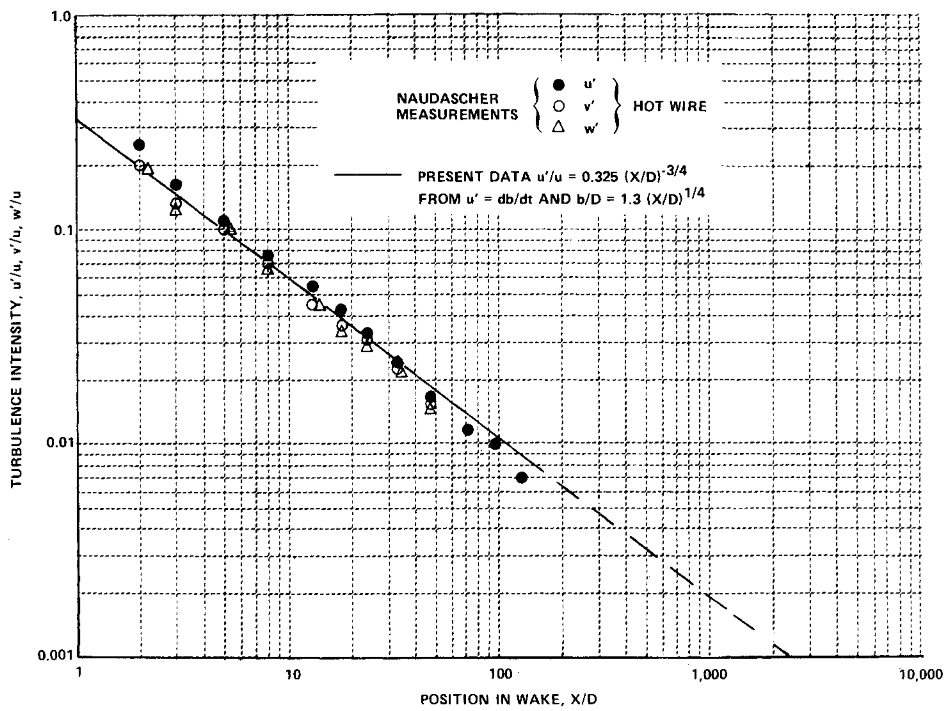

Fig. 3 Decay of turbulence intensity (unstratified flow).

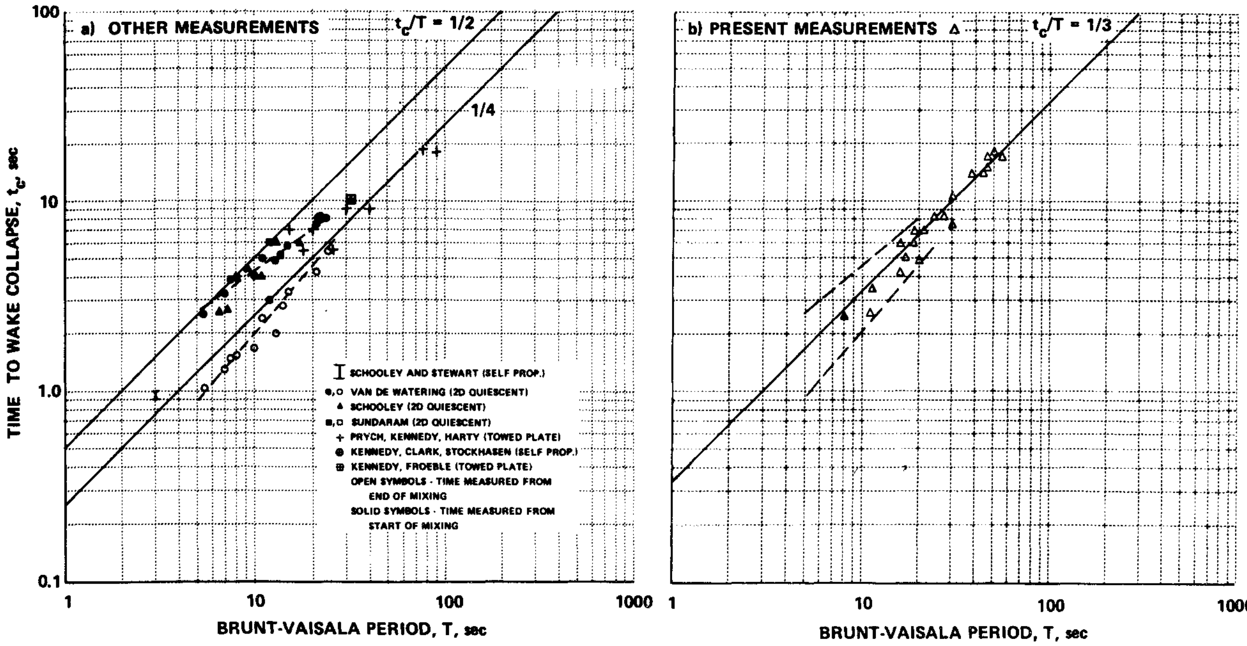

Fig. 4 Correlation of wake collapse time with Brunt-Vaisala period.

4. Wake Growth and Collapse in Stratified Flow

4.1 Wake Collapse Time

The wake of a body travelling through an environment that is stably stratified can be profoundly influenced by the density gradient. Immediately downstream of the vehicle, the wake, which may contain fluid of nearly constant density, will grow at the same rate in all directions, as in the unstratified flow case, (Fig. 2). However, as the turbulent energy of the wake decays with increasing distance from the body (Fig. 3), the restoring action of buoyancy begins to inhibit the vertical expansion of the wake and at the same time enhances the horizontal growth. At some point behind the body, the wake reaches a maximum vertical size followed by a collapse as the fluid returns under the action of gravity to the level at which its density is the same as the environment.

Many investigations,[1-13] primarily two-dimensional unsteady ones, have been directed toward determining the distance or time from the body to wake collapse. A parameter used to characterize the phenomenon is the Brunt-Vaisala period T defined by

$$T = \frac{2\pi}{[(g/\rho)(\partial\rho/\partial z)]^{1/2}} \tag{2}$$

where $\partial\rho/\partial z$ is the ambient density gradient.

Since T is the period at which a parcel of displaced fluid oscillates about its equilibrium-density position, it would be reasonable to expect that the time to wake collapse t_c would be a function of T. However, it must be recognized that disagreement between results from different investigators can be attributed to the fact that Van de Watering[5,6] expresses the Brunt-Vaisala period per radian and Schooley[7,8] per cycle. In addition Schooley measures the time to collapse t_c from the start of mixing and Van de Watering from the end of mixing. As shown in Fig. 4a, this can seriously influence the results since typical values for T are between 5 and 24 sec/cycle with a typical mixing time of 3 sec.

When allowance is made for the different definitions of T and zero time in Fig. 4a, t_c/T from the two-dimensional unsteady experiments is observed to lie between $\frac{1}{4}$ and $\frac{1}{2}$ at low values for T where the mixing time is equal to or greater than t_c. The mixing time becomes a decreasing fraction of the collapse time with increasing T, and the ratio t_c/T approaches a value of $\frac{1}{3}$ at $T \approx 24$ sec/cycle, the upper limit of the measurements. For Schooley and Stewart's experiments[1] with a self-propelled model in which the mixing time is very small relative to the Brunt-Vaisala period T (which itself is only 3 sec), t_c/T in Fig. 4a equals $\frac{1}{3}$. Measurements reported[3,4] for towed plates are also shown to give approximately the same value for the ratio t_c/T in Fig. 4a.

In the present series of experiments, the wake grows downstream of an oscillating grid that is operated continuously in a flow of thermally-stratified water as depicted in Fig. 1. Possible influences of mixing time are eliminated and the collapse of the wake is not constrained by the presence of the mixer in the center as in the two-dimensional unsteady experiments. Measurements were performed for Brunt-Vaisala periods between 8 and 60 extending the range in T higher. The results shown in Fig. 4b indicate that the collapse time is roughly one-third the Brunt-Vaisala period over the range of T studied. It should be emphasized that the collapse time in all the experiments can be estimated only approximately since the arresting of vertical wake growth followed by the collapse is a gradual process spread out over a considerable period of time.

4.2 Analysis of Wake Collapse

To supply guidelines for correlating measurements of the size of the wake generated by a self-propelled body travelling through a stratified medium, a parametric analysis of wake collapse was carried out. In the vicinity of collapse, the turbulent energy in the wake (which makes it grow) is assumed equal to the potential energy (which makes it collapse).

For a wake of rectangular cross section with a linear density gradient, the potential energy can be approximated by $b_h \rho(\alpha-\beta)b_v^3/12$ and the kinetic energy by $b_h \rho(u')^2 b_v/2$ where b_v is the vertical height of the wake, b_h is the horizontal width, ρ is the ambient density, u' is the turbulent velocity fluctuation, g is the acceleration due to gravity, $\alpha = (1/\rho)(\partial\rho/\partial z)|_a$ is the ambient density gradient and $\beta = (1/\rho)(\partial\rho/\partial z)|_w$ is the density gradient in the wake caused by the mixing. If the wake were completely mixed, the wake would have a constant density and β would equal zero. The assumption of a rectangular wake will be shown later to be reasonable.

The representation of the potential and kinetic energies by the previous expressions should be regarded as giving a parametric rather than an exact dependence. Equating the two approximate equations for the kinetic and potential energies yields the following relation for the vertical height of the wake

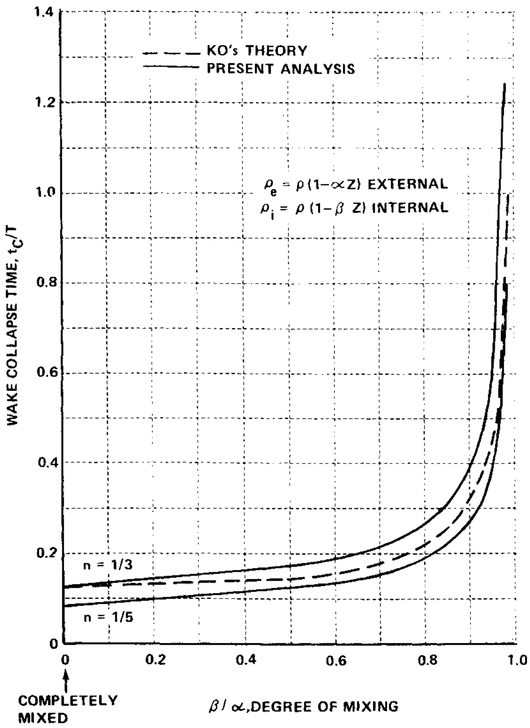

Fig. 5 Effect of mixing on wake collapse time.

$$b_v^2 \approx \frac{6(u')^2}{g(\alpha-\beta)} \tag{3}$$

From measurements (Fig. 3), it has been established that for momentumless wakes in unstratified flow the turbulent intensity u' can be equated to the wake growth rate by $u' \approx db_\infty/dt$. Combining this relation with Eq. (1), and taking $x = ut$ where u is the vehicle velocity, yields

$$u'/b_\infty = n/t \tag{4}$$

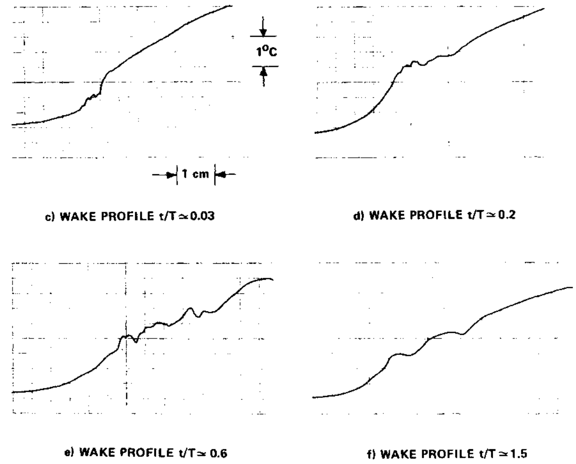

Fig. 6 Measured temperature profiles $T \sim 16$.

Combining Eqs. (4, 3, and 2) gives

$$\frac{b_v}{b_\infty} \approx \frac{0.40n}{(t/T)[1-(\beta/\alpha)]^{1/2}} \tag{5}$$

In other words, the ratio of the vertical height b_v of the wake under stratified flow conditions to the diameter b_∞ when the flow is not stratified can be expressed as a simple function of t/T (the ratio of time $t = x/u$ between the body and the wake location to T the Brunt-Vaisala period). The degree of mixing in the wake enters in the term $[1-(\beta/\alpha)]^{1/2}$.

A physical understanding of the t/T scaling can be obtained by noting that interactions involving flow stratification are often characterized by the local Richardson number Ri which is defined below

$$Ri = [b_\infty{}^2/(u')^2](g/\rho)(\partial\rho/\partial z) \tag{6}$$

The parameter t/T is directly related to Ri since Eq. (6) leads to

$$Ri = [(2\pi/n)(t/T)]^2 \tag{7}$$

Essentially the Richardson number, Ri, represents the ratio of the potential energy to the turbulent energy. Using Eq. (7) in Eq. (5) reveals that $b_v/b_\infty \approx 0.8\pi/(Ri)^{1/2}(1-\beta/\alpha)^{1/2}$. Hence, as physically anticipated, an increase in potential energy (higher Richardson number) decreases the vertical extent of the wake while an increase in turbulence (lower Richardson number) increases the vertical wake size.

4.3 Measurements of Degree of Mixing in Wake

An important parameter revealed in the analysis of wake collapse, Sec. 4.2, is the degree of mixing in the wake, β/α. The effect of mixing on the time to wake collapse t_c can be determined by taking $b_v/b_\infty = 1.0$ in Eq. (5). This yields

$$\frac{t_c}{T} = \frac{0.40n}{[1-(\beta/\alpha)]^{1/2}} \tag{8}$$

In Fig. 5, the ratio t_c/T is plotted against the degree of mixing β/α for n equal to $\frac{1}{3}$ and $\frac{1}{5}$. For the completely mixed case, ($\beta/\alpha = 0$, and $n = \frac{1}{4}$), $t_c/T \approx 0.10$ whereas for the slightly mixed case, ($\beta/\alpha = 0.9$), $t_c/T \approx 0.32$. This is the value for t_c/T observed in Fig. 4 that best correlates the available experimental data on wake collapse time. As shown in Fig. 5, an almost identical variation in t_c/T with β/α has been observed in the computer results from Ko's[16] theoretical analysis in which the value of $\beta/\alpha = 0.9$ also was observed to best fit the measurements.

To determine the degree of mixing in the wake as a function of downstream distance, measurements of the vertical temperature profiles both inside and outside of the wake have been obtained using small thermistor beads that traverse the flow. The results are presented in Fig. 6 for a Brunt-Vaisala period of 16. The ambient profile given in Fig. 6a shows a temperature gradient of roughly $\frac{3}{4}$° C/cm. In Fig. 6b, a temperature probe at a fixed vertical position reveals temperature fluctuations in the wake of about 0.1°C compared with the smooth profile obtained

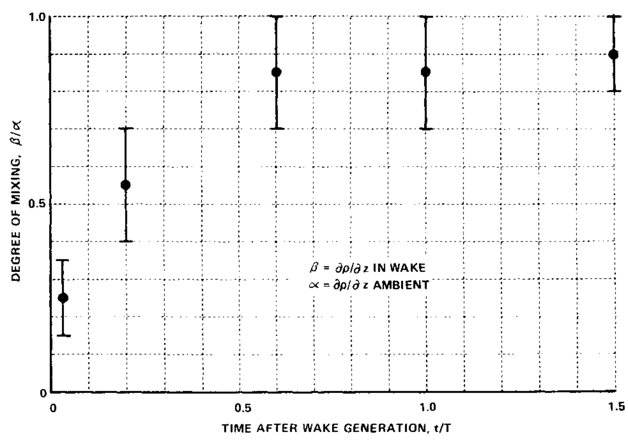

Fig. 7 Measured wake degree of mixing, $T \sim 16$.

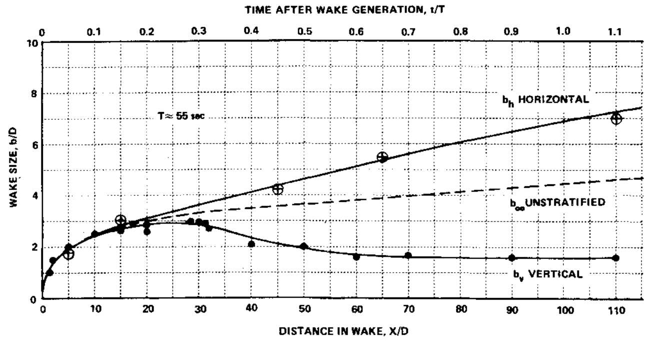

Fig. 8 Measured wake growth.

in the undisturbed stream when the grid is not oscillated. This fluctuation is equivalent to roughly 10–20% of the ambient temperature difference across the wake at this point. The probe is located just inside the upper edge of the wake and the drop in temperature shown in Fig. 6b when the grid is oscillated confirms that the water is being mixed. This can be seen in the wake temperature profile in Fig. 6c in which a wake of nearly constant temperature has been generated near the grid. As shown in Figs. 6d–f, the wake temperature profile returns closer to the ambient one with increasing distance downstream of the grid although large scale motions hinting at interval waves are observed.

From measured profiles such as those given in Fig. 6, the degree of mixing β/α in the wake was calculated. The results are presented in Fig. 7 as a function of time after wake generation. Although considerable uncertainty exists, it is apparent that by the start of collapse at about $t/T \simeq \frac{1}{3}$ (Fig. 4), β/α is approaching a value not much less than unity. This is consistent with analytical observations discussed in Sec. 4.5 requiring a value of $\beta/\alpha = 0.9$ to fit theory to experiment.

4.4 Measurements of Wake Collapse

Using the experimental configuration illustrated in Fig. 1 and described in Sec. 2, measurements of wake growth and collapse were obtained for Brunt-Vaisala periods up to 60. This compares with a value of 3 for the self-propelled model experiments[1] and 6 to 24 for the two-dimensional unsteady experiments[5-13] that are influenced by the mixing time and the presence of the mixer at the center of the turbulent region. The present experiment was run continuously with the temperature gradient allowed to decrease gradually with increasing time so that T varied from about 8 sec/cycle at the start of the experiment to 60 at the end, several hours later. During this period, the wake collapse was observed to move downstream as predicted by the relation $t_c/T \simeq \frac{1}{3}$.

Measurements of the vertical and horizontal growth of the wake, obtained using the pH-color change technique and the experimental configuration shown in Fig. 1, are presented in Fig. 8 for a Brunt-Vaisala period of 55. The measurements are normalized by D, the initial wake diameter which is approximately equal to the grid diameter. From the grid to an x/D of 15 which corresponds to a value for t/T of 0.15, the vertical b_v and horizontal b_h size of the wake are approximately equal to the diameter b_∞ measured in the unstratified flow case, Fig. 2. Further downstream, the vertical size of the wake reaches a maximum of about three diameters and then decreases as the wake collapses. This causes the horizontal extent of the wake to grow at a faster rate than observed in the unstratified flow case. The vertical size of the wake appears to reach an asymptote of roughly 1.7 times the initial diameter by a time of about one Brunt-Vaisala period.

The cross-sectional area of the wake calculated from the measurements in Fig. 8 is shown in Fig. 9 as a function of time after wake generation. The wake area A_w is normalized by the initial area of the wake A_D. For unstratified flow, the circular wake grows according to the relation $A_w/A_D = (1.3)^2(ut/D)^{1/2}$ obtained from the measurements in Fig. 2. For stratification, an assumption must be made concerning the shape of the wake. Ko[16] assumes an elliptical wake. In Fig. 9, this assumption is shown to lead to an unrealistic decrease in wake area just after collapse at $t/T = \frac{1}{3}$. The same behavior is observed when the wake area is calculated for measurements reported by other investigators.[1-9] A rectangular wake will give values for A_w which are too high at collapse but which will likely be realistic at later time.

The most probable variation of wake area with time is shown by the dotted line in Fig. 9. The initially circular wake starts to collapse in the form of an ellipse but quickly approaches a rectangular cross section. Ko's[16] assumption of an ellipse after collapse could account in part for the fact that his analysis predicts horizontal wake dimensions somewhat greater than those measured at long times after collapse.

4.5 Correlation of Measurements of Wake Collapse

4.5.1 *Vertical height*

In the analysis of wake collapse described in Sec. 4.2, it was established that the ratio of the vertical wake size to the diameter

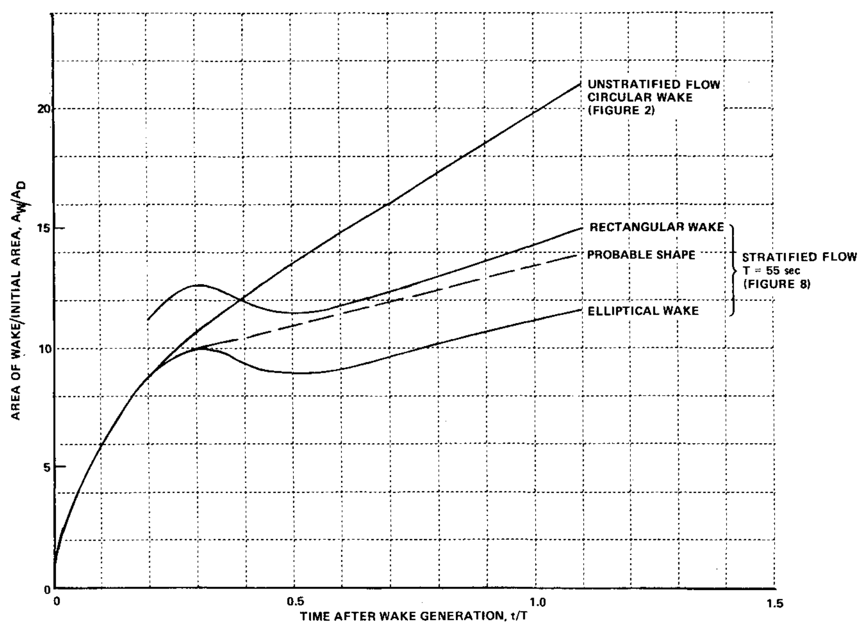

Fig. 9 Growth of wake area.

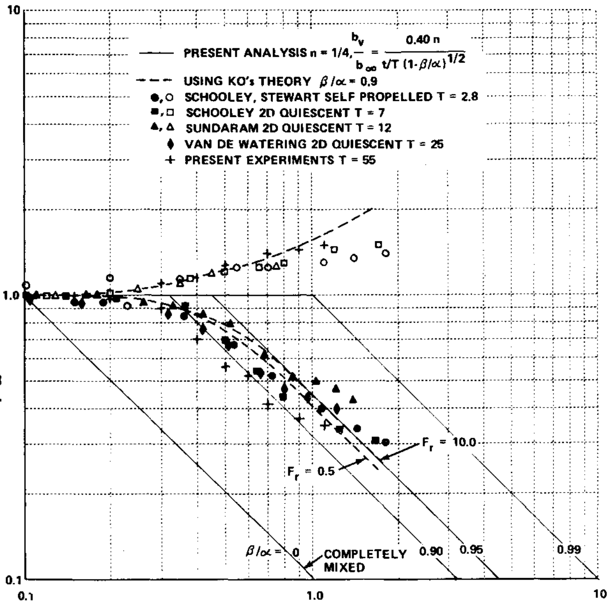

Fig. 10 Correlation of wake dimension during collapse.

in unstratified flow, b_v/b_∞ was inversely proportional to t/T, the ratio of the time after wake generation (x/u) to the Brunt-Vaisala period. The present measurements from Fig. 8 of wake dimensions during collapse for $T = 55$ as well as those reported by four different investigators[1,6-9] covering a range of Brunt-Vaisala period from 3 to 24 are correlated well when plotted in Fig. 10 in the form b_v/b_∞ against t/T. The results include measurements from self-propelled model tests, two-dimensional unsteady experiments and the present experimental configuration shown in Fig. 1.

Curves of b_v/b_∞ vs t/T obtained from Eq. (5) are plotted in Fig. 10 for different values of β/α, the ratio of the density gradient in the wake to the ambient gradient. Comparison with the experimental data requires a value for the degree of mixing β/α of roughly 0.90–0.95. This is consistent with the measurements of β/α (Figs. 6 and 7) which indicate that the wakes studied in the lab are only slightly mixed at collapse. In Ko's[16] investigation, curves are plotted for $\beta/\alpha = 0.9$ showing wake growth and collapse for Froude numbers between 10 and $\frac{1}{2}$ which correspond to Richardson numbers defined by Eq. (6) between 10^{-2} and 4.0. With the present analysis, Ko's curves are nearly collapsed in Fig. 10 into one curve falling through the experimental data. As a result, the prediction of wake dimensions during collapse can be considerably simplified by use of the b_v/b_∞ vs t/T scaling.

The observation that the degree of mixing in the wakes is slight, $\beta/\alpha \simeq 0.9$ to 0.95, needs to be qualified somewhat. In all the measurements, such as those in Figs. 8 and 10, stratification affects the initial growth rate only slightly so that $b_v/b_\infty \approx 0.9$ to 1.0 before collapse at $t/T \simeq \frac{1}{3}$. The density gradient inside the wake at this time and not that at $t = 0$ determines the wake collapse. From Fig. 9, it is seen that the wake area at $t/T = \frac{1}{3}$ is roughly ten times the initial area. At $t = 0$, the wake may well be completely mixed with $\beta/\alpha = 0$. However, with growth, the entrainment of such a large quantity of ambient fluid into the wake would be expected to produce a density gradient at collapse not much different from the ambient gradient, as observed.

To estimate the vertical extent of a wake during the collapse phase, Eq. (5) can be combined with the measured growth rate in unstratified flow $b_\infty/D = K(x/D)^n$ to yield

$$\frac{b_v}{D}\left(\frac{1}{F}\right) = \frac{0.40Kn}{[1-(\beta/\alpha)]^{1/2}}\left(\frac{x}{D}\right)^{n-1} \tag{9}$$

where $F = uT/D$ is a Froude number defined using the vehicle velocity u, the ambient Brunt-Vaisala period T, and the initial wake diameter D. Comparison with Ko's[16] definition of a Froude number $Fr = u_r T/2\pi D$ where u_r, the initial turbulent intensity is taken by Ko to be equal to $0.25u$, gives a relation between the two Froude numbers of $F \approx 25Fr$.

Following Eq. (9), $(b_v/D)(1/F)$ is plotted against x/D in Fig. 11. For comparison, Ko's[16] theoretical curves from wake collapse on downstream for Fr of 0.5, 1.0, and 10 are also given in the figure. When plotted in this fashion, Ko's results for different Froude numbers fall into one curve given by $b_v/D(F) = 0.57(x/D)^{-4/5}$. For the present analysis, as shown in Fig. 11, this corresponds to Eq. (9) with $K = 1.3$, $\beta/\alpha = 0.94$, and $n = \frac{1}{5}$.

A simple scaling law for the post collapse behavior of the wake is thus obtained from the two analyses when the vertical size of the wake is divided by the Froude number, F, which is proportional to the ratio of the turbulence energy to the potential energy. In physical terms, at a given wake position x/D, the vertical extent of the wake, b_v/D, which is directly proportional to the Froude number, F, increases as the wake turbulence increases with velocity and decreases as the potential energy (D/T) increases. In the use of Eq. (9) just described, there is some uncertainty about the values for K and n while the effect of the degree of mixing in the wake needs to be investigated further.

In Fig. 12, a comparison is made between predictions of the vertical dimensions of the wake using Ko's[16] computer analysis

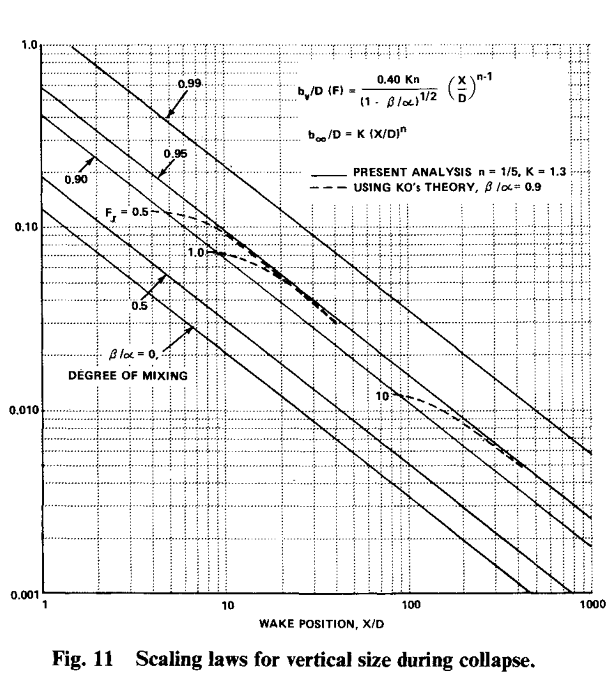

Fig. 11 Scaling laws for vertical size during collapse.

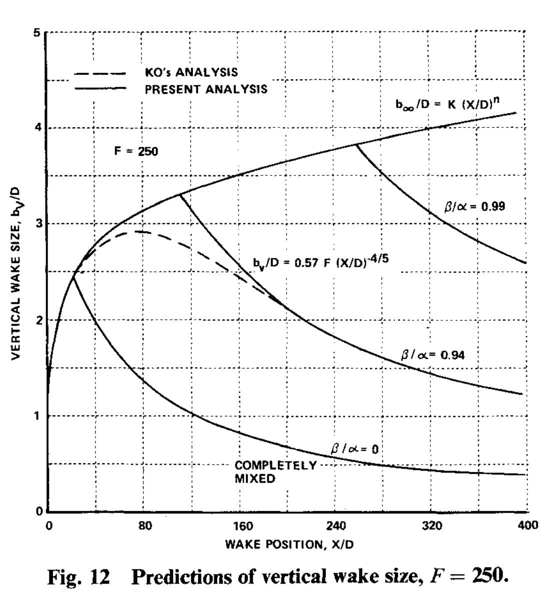

Fig. 12 Predictions of vertical wake size, $F = 250$.

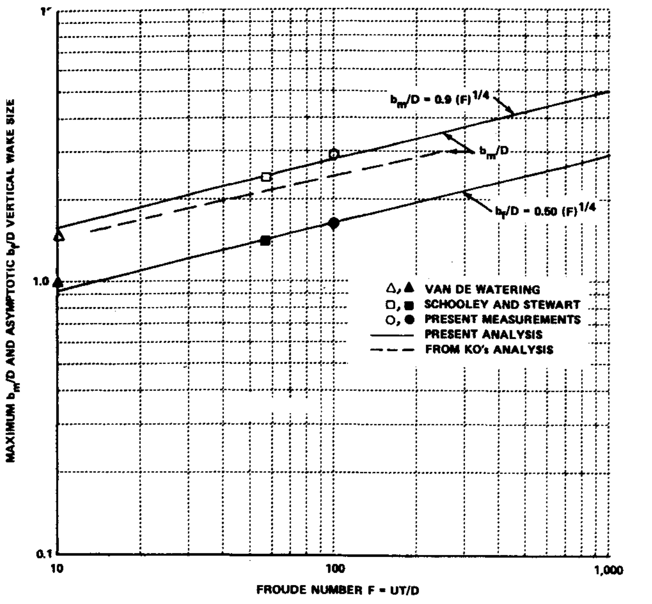

Fig. 13 Correlation of maximum and asymptotic vertical wake sizes.

and the power law growth rates developed in the present paper. The vertical size is plotted against wake position x/D for $F = 250$. The Froude number F is inversely proportional to the square root of the Richardson number. The curve for unstratified flow, $b_\infty/D = K(x/D)^n$ plotted in Fig. 12 gives the extent of the wake up to collapse. A family of curves obtained from Fig. 11 for different degrees of mixing β/α gives the size of the wake after collapse. Comparing the $\beta/\alpha = 0.94$ curve with Ko's[11] analysis, shown by the dotted line in Fig. 12, excellent agreement is obtained except in the immediate vicinity of collapse. At the intersection of the unstratified flow and post collapse curves, a maximum difference of roughly 12% is observed. Clearly, a fairing of the two power-law curves would give closer agreement with Ko's result in this region. The important point is that both analyses give the same decay rate of vertical size with distance in the wake.

To estimate the size of a wake in stratified flow, it is convenient to express the dimensions in terms of the time after wake generation. Before collapse, the growth of the circular wake given by $b_\infty/D \approx K(x/D)^n$ can be expressed in the form

$$b_\infty/D = K(F)^n(t/T)^n = 1.3(F)^{1/4}(t/T)^{1/4} \qquad (10)$$

Combination of Eq. (10) with the analytical estimate of the vertical extent of the wake during collapse given by Eq. (5) and confirmed by the experimental data in Fig. 10 yields an expression for b_v/D in terms of t/T

$$\frac{b_v}{D} = \frac{0.40nK(F)^n}{(1-\beta/\alpha)^{1/2}}\left(\frac{t}{T}\right)^{n-1} \qquad (11)$$

Equation (11) gives the vertical wake height normalized by the initial wake diameter as a function of the initial Froude number, the degree of mixing in the wake, two parameters n and K determined from the wake growth rate in unstratified flow, and the time after generation normalized by the ambient Brunt-Vaisala period. Substitution of the experimental values of $K = 1.3$, $n = \frac{1}{4}$ (Fig. 2) and $\beta/\alpha \approx 0.93$ (Fig. 10) into Eq. (11) yields

$$b_v/D = 0.5(F)^{1/4}(t/T)^{-3/4} \qquad (12)$$

4.5.2 Vertical asymptote and maximum extent

The vertical size of the wake can be estimated using Eq. (12) between collapse and $t/T \approx 1.0$ at which point measurements such as those in Figs. 8 and 10 indicate that the wake vertical size may reach an asymptote b_f. Putting $t/T = 1.0$ in Eq. (12) gives

$$b_f/D \simeq 0.5(F)^{1/4} \qquad (13)$$

Equation (13), which requires further verification, gives a relation for the asymptotic vertical wake size b_f in terms of the initial size and Froude number. For the conditions of the present experiment, $u = 1.1$ cm/sec, $D = 0.6$ cm, and $T = 55$ sec, $F = UT/D = 100$. As shown in Fig. 13, Eq. (13) gives an asymptotic wake height b_f/D of 1.6 compared with a measured value of 1.7 in Fig. 8. For Schooley and Stewart's[1] self-propelled model experiments in which $u = 45$ cm/sec, $D = 2.2$ cm, and $T = 2.8$ sec, Eq. (13) gives a value of 1.4 for b_f/D equal to that measured.

A similar relation can be obtained for the maximum vertical extent b_m of the wake by substituting $t_c/T \approx \frac{1}{3}$ from Fig. 4 into Eq. (10) and noting from Fig. 10 that $b_m/b_\infty \approx 0.9$ to give

$$b_m/D \approx 0.9(F)^{1/4} \qquad (14)$$

For the present experiment, Eq. (14) as shown in Fig. 13 gives $b_m/D \approx 2.8$ compared with a measured value of 2.9 while for Schooley and Stewart's[1] tests the measured and estimated values are both 2.4.

Ko's[16] analysis is shown in Fig. 13 to give the same variation of b_m/D with Froude number, F, although the values of b_m/D are slightly lower than those measured. In addition, Ko's[16] analysis predicts values of b_f/D less than those measured in the region where an asymptote appears to be reached. This could account in part for Ko's analysis overpredicting the horizontal extent of the wake in this region.

Van de Watering et al.[6] report measurements of b_m/D and b_f/D in terms of the initial wake growth rate. Use of Ko's assumptions that $u'/u = 0.25$ initially for Van de Watering's measurements gives values of (F) between 2 and 10. Van de Watering's measurements also yield the $(F)^{1/4}$ variation given in Fig. 11 in which his measurements at the highest value of $F = 10$ are shown to agree well with Eqs. (13) and (14).

4.5.3 Horizontal width

As the wake collapses vertically, the horizontal growth rate is enhanced so that the horizontal width b_h becomes greater than the diameter in unstratified flow b_∞. The ratio b_h/b_∞ from the present as well as other investigations[1,6–9] is correlated quite well in Fig. 10 by the parameter t/T and reaches a value of about 1.6 at $t/T = 2$ compared with the vertical size $b_v/b_\infty \approx \frac{1}{3}$. In other words, the wake width at this point is roughly five times the wake height. Ko's theory[16] is shown in Fig. 10 to overpredict the horizontal wake size for t/T greater than about 0.7. This is probably because of Ko's assumption of an elliptical wake, although the presence of the sidewalls may have restricted the horizontal growth in some of the previous experiments.[1]

To predict the horizontal extent of a wake during collapse, use can be made of the correlation of measurements in Fig. 10. The data can be fitted by

$$b_h/b_\infty \approx 1.55(t/T)^{1/4} \qquad (15)$$

This expression when combined with the unstratified growth rate, Eq. (10), gives for $K = 1.3$ and $n = \frac{1}{4}$

$$b_h/D \approx 2.0(F)^{1/4}(t/T)^{1/2} \qquad (16)$$

Equation (16) gives the horizontal enhancement of the wake growth rate during the period when the wake is collapsing vertically. If, as postulated, a vertical asymptote is reached at about $t/T \approx 1.0$, it is reasonable to expect that the horizontal growth rate would be reduced. In this region, the wake width b_c is assumed to increase at the same rate as in unstratified flow with $n = \frac{1}{4}$ but with $K = 2.0$ because Eq. (16) must be matched at $t/T = 1.0$. This gives

$$b_c/D = 2.0(F)^{1/4}(t/T)^{1/4} \qquad (17)$$

4.5.4 Combination of estimates of wake dimensions

The lab measurements of grid generated and model wakes yield means for predicting the effect of stratification on wake growth for various conditions through the use of Eq. (10) up to collapse ($t_c/T \approx \frac{1}{3}$) and Eqs. (12–17) downstream of collapse. With the best values presently available for the uncertain parameters used in those equations ($K = 1.3$, $n = \frac{1}{4}$, $\beta/\alpha = 0.93$), it is possible to estimate wake dimensions for different speeds, initial

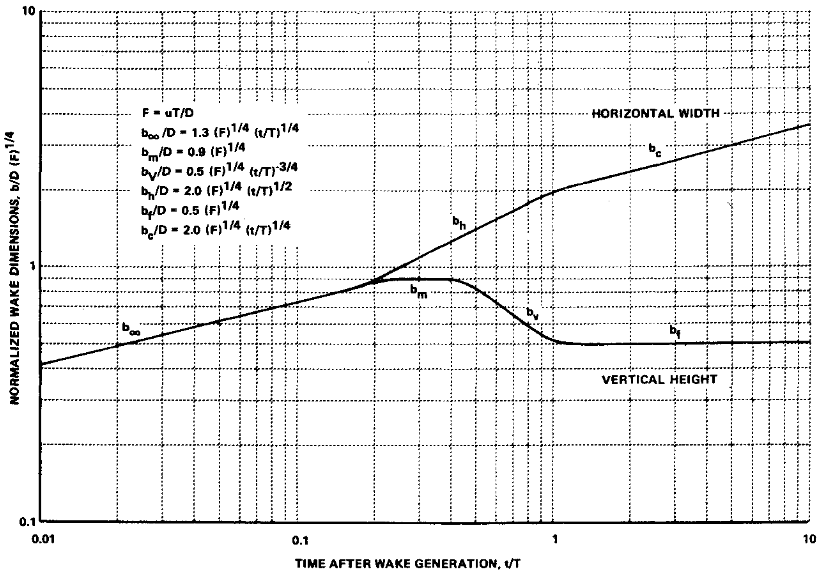

Fig. 14 Scaling laws for wake growth and collapse.

wake sizes, and ambient stratification. In addition, the effect of varying the parameters K, n, and β/α determined from theory and experiment could be calculated.

Referring to Eqs. (12) and (16), it is observed that both the vertical, b_v, and horizontal, b_h, dimensions of the wake vary weakly with the velocity u being proportional to $u^{1/4}$. As a consequence, measurements of wake dimensions at equal times after generation would not change very much with velocity. On the other hand, the vertical extent of the wake during collapse is very sensitive to the ambient stratification being directly proportional to the Brunt-Vaisala period T, although the asymptotic height only varies with $T^{1/4}$.

The scaling relations established in this section from experimental data for wake growth and collapse reveal that the important parameters are the Froude number, F, and the ratio t/T of the residence time in the wake to the Brunt-Vaisala period. This is demonstrated in Fig. 14 where the wake dimensions b/D divided by $(F)^{1/4}$ are plotted against t/T. In this manner, two unique curves are obtained for estimating the horizontal width and vertical height of a wake in stratified flow. Verification of these curves or similar ones possibly modified by different values for the constants and exponents must rest ultimately on comparison with experimental data at higher Froude numbers.

From the equations and curves for wake growth in Fig. 14, unique curves could also be determined for the wake area by plotting $A_w/A_D(F)^{1/2}$, the wake area normalized by the initial area and divided by the square root of the Froude number, against the time after wake generation. The results show that when the flow is stratified, the inhibition of vertical entrainment after collapse reduces the wake area to approximately one half to one third the area anticipated for unstratified flow. The assumption of an elliptical wake during collapse leads to an unrealistic decrease in wake area with time. This can be avoided by assuming that the wake becomes nearly rectangular in shape soon after collapse.

5. Conclusions

Techniques have been developed in the paper for giving preliminary estimates of the horizontal width and vertical height of the wake of a self-propelled vehicle moving through a stratified body of water. Use was made of measurements reported by other investigators as well as those of the present investigation in which a wake was generated by oscillating a grid in a stably-stratified flow. A pH-sensitive indicator yielded a neutrally-buoyant tracer which visualized wake growth and vertical collapse. In addition, measurements were made of the velocity and temperature profiles in the simulated wake.

The measured velocity profiles indicated that the wake was nearly momentumless. The temperature profiles showed the change from an initial nearly constant-temperature wake to one with a temperature profile not much different from the ambient at the start of collapse. Over a wide range of Brunt-Vaisala periods, as well as for several different experimental configurations, the maximum vertical extent of the wake, which defines the start of collapse, occurred at a time after wake generation of roughly one third the Brunt-Vaisala period. Up to collapse, it was observed that the wake growth over a range of conditions could be represented quite well by the expression developed for unstratified flow. At collapse, the area of the wake was typically ten times the initial area. It is therefore not surprising that the entrainment of this large quantity of ambient fluid into the wake would produce the observed density gradient at collapse nearly equal to the ambient gradient.

The observation of nearly equal density gradients means that the wake potential energy at collapse is comparatively low so that the subsequent vertical collapse of the wake is a gradual process. A parametric analysis of wake collapse was performed using power laws established for wake growth and turbulence intensity decay. The analysis revealed that the ratio of the vertical height of the wake under stratified flow conditions to the diameter when the flow is not stratified could be expressed as a simple function of the ratio of the time after wake generation to the Brunt-Vaisala period. The various measurements reported in the literature as well as those of the present investigation were correlated well when plotted in this fashion. In addition, the theoretical curves developed by Ko[16] for different Froude numbers were nearly collapsed into one curve passing through the experimental data in agreement with results of the present analysis.

From the observed correlation of the vertical extent of the wake as well as others developed for the maximum, asymptotic and horizontal wake dimensions, two curves were established for estimating the horizontal and vertical size of a wake in stratified flow. In essence, at any particular time after wake generation, expressed in terms of the Brunt-Vaisala period, a fixed value exists for each of the ratios of the vertical and horizontal wake sizes to the initial wake diameter multiplied by the Froude number raised to the minus one-quarter power. The Froude number is defined as the product of the velocity and the Brunt-Vaisala period divided by the initial wake size. Verification of these relations, possibly modified by different values for the constants and exponents, must rest ultimately on comparison with data from larger scale experiments. Some areas for investigations include the initial wake size, the degree of mixing, the effect of stratification on turbulence intensities and the asymptotic behavior of the wake.

References

[1] Schooley, A. H. and Stewart, R. W., "Experiments with a Self-Propelled Body Submerged in a Fluid with a Vertical Density Gradient," *Journal of Fluid Mechanics*, Vol. 15, Pt. I, 1963, p. 83.

[2] Stockhausen, P. J., Clark, C. B., and Kennedy, J. F., "Three-Dimensional Momentumless Wakes in Density-Stratified Liquids," Rept. 93, 1966, MIT Hydrodynamics Lab., Cambridge, Mass.

[3] Prych, E. A., Harty, F. R., and Kennedy, J. F., "Turbulent Wakes in Density-Stratified Fluids of Finite Extent," TR 65, July 1964, Hydrodynamics Lab., MIT, Cambridge, Mass.

[4] Kennedy, J. F. and Froebel, R. A., "Two-Dimensional Turbulent Wakes in Density-Stratified Liquids," ASME Publication, 64-WA/UNT-11, 1964, New York.

[5] Van de Watering, W. P. M., "The Growth of a Turbulent Wake in a Density-Stratified Fluid," Rept. 231-12, 1966, Hydronautics, Inc., Laurel, Md.

[6] Van de Watering, W. P. M., Tulin, M. P., and Wu, J., "Experiments on Turbulent Wakes in a Stable Density-Stratified Environment," TR 231-24, 1969, Hydronautics, Inc., Laurel, Md.

[7] Schooley, A. H., "Wake Collapse in a Stratified Fluid," *Science*, Vol. 157, 1967, p. 421.

[8] Schooley, A. H., "Wake Collapse in a Stratified Fluid: Experimental Exploration of Scaling Characteristics," *Science*, Vol. 160, 1968, p. 763.

[9] Sundaram, T. R., Stratton, J., and Rehm, R. G., "Turbulent Wakes in a Stratified Medium," Rept. AG-3018-A-1, Nov. 1971, Calspan Corp., Buffalo, N.Y.

[10] Wu, J., "Collapse of Turbulent Wakes in Density Stratified Media," Rept. 231-4, 1965, Hydronautics, Inc., Laurel, Md.

[11] Wu, J., "Flow Phenomena Cause by the Collapse of a Mixed Region in a Density-Stratified Medium," Rept. 231-11, 1966, Hydronautics, Inc., Laurel, Md.

[12] Wu, J., "Wake Collapse and Subsequent Generation of Internal Waves in a Density-Stratified Medium," Rept. 231-17, 1968, Hydronautics, Inc., Laurel, Md.

[13] Wu, J., "Mixed Region Collapse with Internal Wave Generation in a Density-Stratified Medium," *Journal of Fluid Mechanics*, Vol. 35, Pt. 3, 1969, p. 531.

[14] Wessel, R., "A Numerical Study of the Collapse of a Perturbation in an Infinite, Density-Stratified Fluid," TR 169, 1968, Argonne National Lab., Argonne, Ill.; see also *Physics of Fluids*, Vol. 12, No. 12, 1969, II-171.

[15] Mei, C. C., "Collapse of a Homogeneous Fluid in a Stratified Fluid," *Proceedings of the 12th International Congress of Applied Mechanics*, edited by M. Hetenyi and W. G. Vincenti. Springer-Verlag, New York, 1969.

[16] Ko, D. R. S., "Collapse of a Turbulent Wake in a Stratified Medium," Rept. R-18202-6001-RO, Vol. 2, Nov. 1971, TRW Systems, Redondo Beach, Calif.

[17] Naudascher, E., "Flow in the Wake of Self-Propelled Bodies and Related Sources of Turbulence," *Journal of Fluid Mechanics*, Vol. 2, Pt. 4, 1965, p. 625.

[18] Pao, Y. H., "Turbulence Measurements in Stably Stratified Liquids," D1-82-0959, Feb. 1970, Boeing Scientific Research Lab., Seattle, Wash.

[19] Schooley, A. H. and Hughes, B. A., "An Experimental and Theoretical Study of Internal Waves Generated by the Collapse of a Two-Dimensional Mixed Region in a Density Gradient," *Journal of Fluid Mechanics*, Vol. 51, Pt. 1, 1972, pp. 159–175.

[20] Hartman, R. J. and Lewis, H. W., "Wake Collapse in a Stratified Fluid: Linear Treatment," *Journal of Fluid Mechanics*, Vol. 51, Pt. 3, 1972, pp. 613–618.

[21] Baker, D. J., "A Technique for Precise Measurement of Small Fluid Velocities," *Journal of Fluid Mechanics*, Vol. 26, 1966, p. 573.

[22] Merritt, G. E. and Rudinger, G., "Thermal and Momentum Diffusivity Measurements in a Turbulent Stratified Flow," AIAA Paper 72-80, San Diego, Calif., 1972.

[23] Birkhoff, G. and Zarantonello, E. H., *Jets, Wakes and Cavities*, Academic Press, New York, 1957.

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Laminar Boundary Layer near the Symmetry Plane of a Prolate Spheroid

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The previous investigation of a three-dimensional boundary layer near the plane of symmetry of an inclined spheroid has been extended to provide 1) wider range of solutions, 2) explanations to questions left unanswered before, and 3) comparisons with experiments. Extended solutions provide more complete trends of the boundary-layer behavior for incidences from 0°–90° and for thickness ratios ranging from unity for a sphere to nearly zero for a long inclined cylinder. Among these trends is how the separation changes from one type to another and then back again with increasing incidence. Explanations are given for a number of unconventional features, including the reversal of the lateral derivative of the cross velocity profile and the flattening of the longitudinal velocity profile. The latter has long been known as a two-dimensional turbulent boundary-layer phenomenon, and it is seen here also to be a three-dimensional phenomenon. Agreement with Wilson's recent experiments (designed specifically for partial check of the results obtained earlier) enhances the significance of the results.

Nomenclature

a = semimajor axis
A = regular boundary-layer region
b = semiminor axis
B = partially reversed region
C = separated region
c_f = skin friction
p = pressure
h_μ = metric coefficient
h_θ = metric coefficient
R = vortex starting point
Re = Reynolds number
S = separation point
u = meridional velocity
U = freestream meridional velocity
v = circumferential velocity
$\partial v/\partial\theta$ = circumferential derivative of v
$\partial V/\partial\theta$ = freestream $\partial v/\partial\theta$
z = normal coordinate
α = incidence angle
μ = meridional coordinate
θ = circumferential coordinate

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